# Unit 4 Topic 1:

orange text → explanation or instruction
black text → working out or answer

Q1) $f(x) = x^3 - 6x^2 + 9x + 5$

a) $f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

b) The first derivative is used the find the instantaneous rate of change of a function.

It can also be thought of as finding the gradient of the tangent at a particular point.

c) To determine change in concavity, need to first find point(s) of inflection by letting second derivative equal 0

$$f''(x) = 6x - 12$$
$$0 = 6x - 12$$
$$6x = 12$$
$$x = 2$$

$\therefore$ Point of inflection (e.g. change in concavity) at $x = 2$

Now find concavity either side of $x = 2$:

$f''(1) = 6(1) - 12$
$= -6$

Negative value $\longrightarrow$ $\therefore$ concave up

$f''(3) = 6(3) - 12$
$= 6$

positive value $\longrightarrow$ $\therefore$ concave down

$\therefore$ concave up for $x < 2$ and concave down for $x > 2$

Q1d) To find point of inflection, let $f''(x) = 0$.

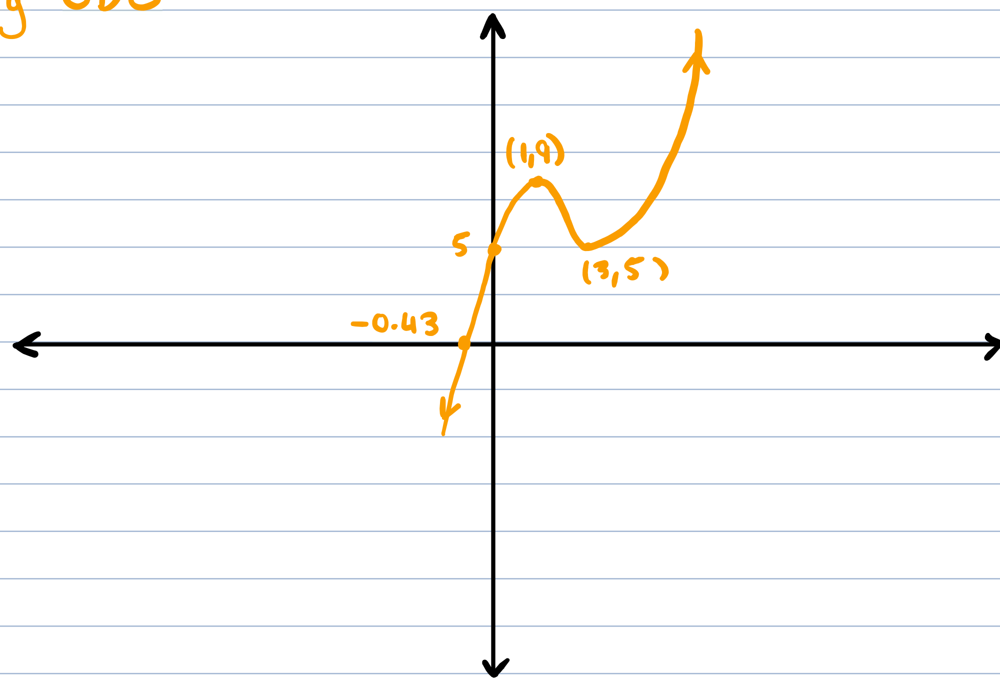
From part c, point of inflection at $x = 2$

Need to find corresponding y-coordinate by substituting $x = 2$ into original function

$$f(2) = (2)^3 - 6(2)^2 + 9(2) + 5$$
$$= 8 - 24 + 18 + 5$$
$$= 7$$

$\therefore$ Point of inflection at $(2, 7)$

e) Using GDC:



Q2) $v(t) = 2t^2 - 4t + 2$

$$\begin{tikzcd}
\text{disp} \arrow[d, bend right, "\text{diff}"'] \\
\text{vel} \arrow[u, bend right, "\text{int}"'] \arrow[d, bend right, "\text{diff}"'] \\
\text{accel} \arrow[u, bend right, "\text{int}"']
\end{tikzcd}$$

a) $a(t) = 4t - 4$

b) $0 = 4t - 4$

$4t = 4$

$\therefore t = 1$

$\therefore$ Acceleration is equal to $0\,m/s^2$ after 1 second

c) Displacement is the integral of velocity

$$s(t) = \int 2t^2 - 4t + 2 \; dt$$

$$s(t) = \frac{2t^3}{3} - \frac{4t^2}{2} + 2t + c$$

To find $c$, substitute $t=0$ and $s(t)=3$

$$3 = 0 - 0 - 0 + c$$

$$\therefore c = 3$$

$$\therefore s(t) = \frac{2t^3}{3} - 2t^2 + 2t + 3$$

d) acceleration is 0 when $t=1$

$\therefore$ Sub $t=1$ into displacement function:

$$s(1) = \frac{2(1)^3}{3} - 2(1)^2 + 2(1) + 3$$

$$= \frac{2}{3} - 2 + 2 + 3$$

$$= \frac{2}{3} + 3$$

$$= \frac{2}{3} + \frac{9}{3}$$

$$= \frac{11}{3} \quad \text{or} \quad \approx 3.67m$$

$\therefore$ Displacement after 1 second is $3.67m$

Q3)

$f(x) = x^4 - 4x^3 \longrightarrow f'(x) = 4x^3 - 12x^2 \longrightarrow f''(x) = 12x^2 - 24x$

To find stationary points, let $f'(x) = 0$ :

$$f'(x) = 4x^3 - 12x^2$$

$$0 = 4x^2(x-3)$$

Using null factor law:

$4x^2 = 0$ or $x - 3 = 0$

$\therefore x = 0$ or $x = 3$

Find y-values of stationary points:

$f(0) = (0)^4 - 4(0)^3 = 0$

$f(3) = (3)^4 - 4(3)^3 = -27$

$(0,0)$ and $(3,-27)$

b) $f''(0) = 12(0)^2 - 24(0) = 0$

Since $f''(x) = 0$

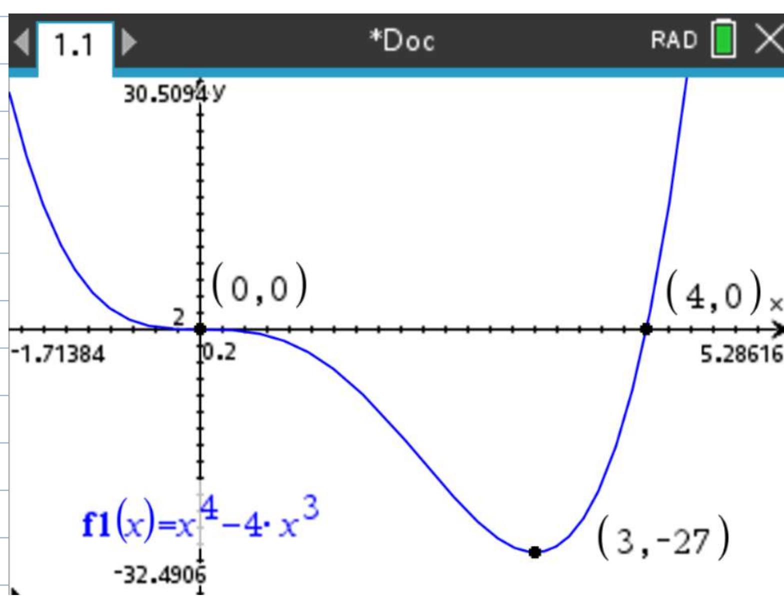
$\therefore$ Point of inflection

$f''(3) = 12(3)^2 - 24(3) = 36$

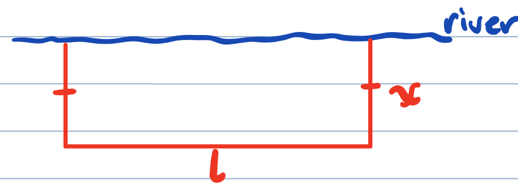
Since $f''(x) > 0$

$\therefore$ Local minimum

c)



Q4) Best approach would be to first draw a diagram:




a) Total perimeter is 120m

$$\therefore P = 2x + l$$
$$120 = 2x + l$$
$$\therefore l = 120 - 2x$$

Substitute $l$ into area formula to optimise area:

$$A = lw$$
$$= (120-2x)x$$
$$A(x) = 120x - 2x^2$$

To find maximum, let $A'(x)=0$ and solve for $x$:

$$A'(x) = 120 - 4x$$
$$0 = 120 - 4x$$
$$4x = 120$$
$$x = 30 \text{ m}$$

c) To justify that the area is a maximum, use second derivative test

$$A''(x) = -4$$

negative value $\rightarrow$ $\therefore$ Maximum

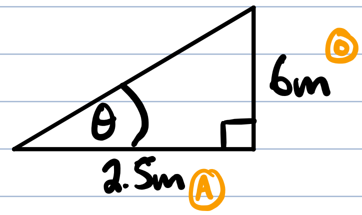
d) To find max area, substitute $x=30$ into $A(x)$:

$$A(30) = 120(30) - 2(30)^2$$
$$= 1800 \text{ m}^2$$

$\therefore$ Max Area is $1800 \text{m}^2$

# Unit 4 Topic 2 :

Q1) a)



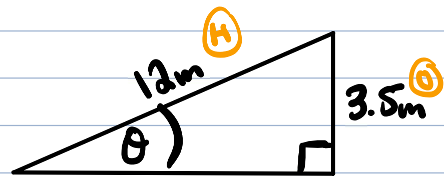

$\tan\theta = \frac{O}{A}$

$\tan\theta = \frac{6}{2.5}$

$\theta = \tan^{-1}\left(\frac{6}{2.5}\right)$

$\theta = 67.38°$

b)



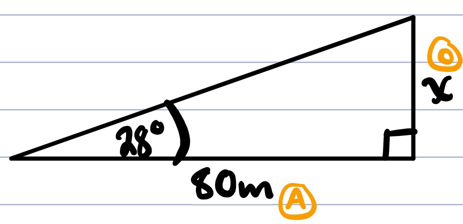

$\sin\theta = \frac{O}{H}$

$\sin\theta = \frac{3.5}{12}$

$\theta = \sin^{-1}\left(\frac{3.5}{12}\right)$

$\theta = 16.95°$

c)




$\tan\theta = \frac{O}{A}$

$\tan 28° = \frac{x}{80}$

$x = 80 \times \tan 28°$

$x = 42.6m$

Q2)

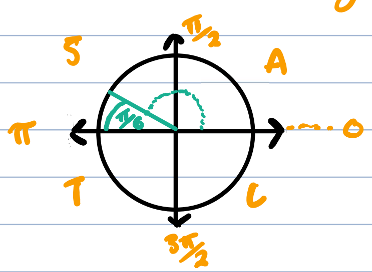
(i) $\sin\left(\frac{\pi}{3}\right)$

Using exact value table:

$$\sin\left(\tfrac{\pi}{3}\right) = \sqrt{3}/2$$

(ii) $\cos\left(\frac{5\pi}{6}\right)$

Using unit circle analysis:




$$\cos\left(\tfrac{5\pi}{6}\right) = -\cos\left(\tfrac{\pi}{6}\right)$$
$$= -\sqrt{3}/2$$

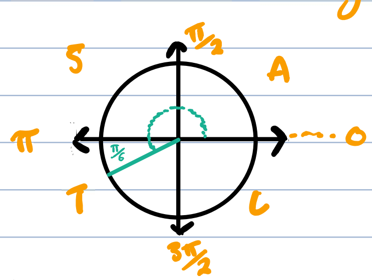
(iii) $\tan\left(\pi/4\right)$

Using exact value table:

$$\tan\left(\pi/4\right) = 1$$

(iv) $\sin\left(7\pi/6\right)$

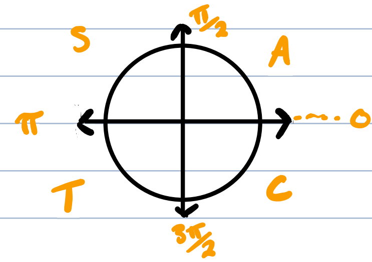
Using unit circle analysis:




$$\sin\left(7\pi/6\right) = -\sin\left(\pi/6\right) = -1/2$$

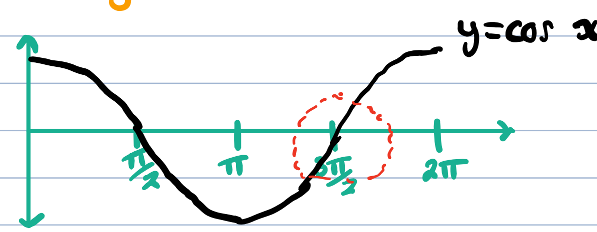
## Exact Value Table

| | $0^\circ$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $\pi/2$ |
|---|---|---|---|---|---|
| $\sin\theta$ | 0 | $1/2$ | $1/\sqrt{2}$ | $\sqrt{3}/2$ | 1 |
| $\cos\theta$ | 1 | $\sqrt{3}/2$ | $1/\sqrt{2}$ | $1/2$ | 0 |
| $\tan\theta$ | 0 | $1/\sqrt{3}$ | 1 | $\sqrt{3}$ | undefined |




(v) $\cos\left(\frac{3\pi}{2}\right)$

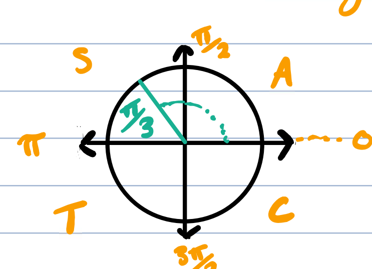
Since multiple of $\frac{\pi}{2}$, use graph analysis:




$$\therefore \cos\left(3\pi/2\right) = 0$$

(vi) $\tan\left(2\pi/3\right)$

Using unit circle analysis:



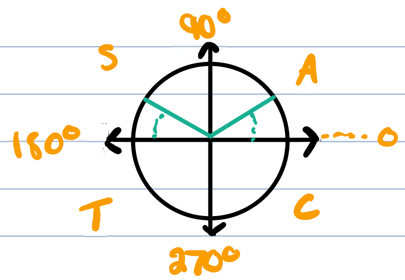

$$\tan\left(\tfrac{2\pi}{3}\right) = -\tan\left(\tfrac{\pi}{3}\right) = -\sqrt{3}$$

Q3) (i) $\sin\theta = \frac{1}{2}$

Using exact value table:

$\theta = 30°$

Use unit circle analysis for second value:

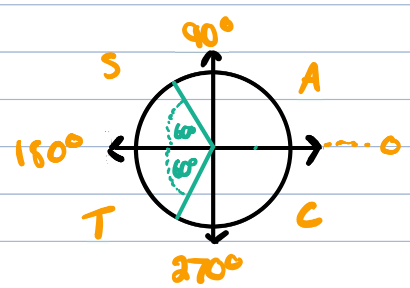



$\therefore \theta = 30°, 150°$

(ii) $\cos\theta = -\frac{1}{2}$

Using exact value table:

$\theta = 60°$, but in quadrants where cos is negative



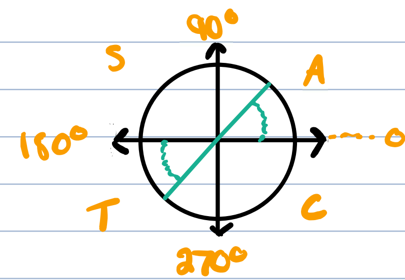

$\therefore \theta = 120°, 240°$

(iii) $\tan\theta = 1$

Using exact value table:

$\theta = 45°$

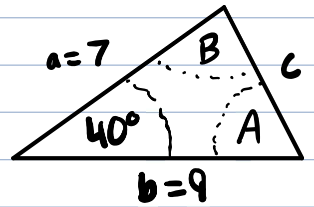
Use unit circle analysis for second value:




$\therefore \theta = 45°, 225°$

Q3)

a)




b) Use cosine rule for side c:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

$$c = \sqrt{7^2 + 9^2 - 2 \times 7 \times 9 \times \cos 40°}$$

$$c \approx 5.79 \text{cm}$$

c) Find ∠A using sine rule:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

Rearranging for A gives:

$$\sin A = \frac{a \cdot \sin C}{c}$$

$$A = \sin^{-1}\left(\frac{a \cdot \sin C}{c}\right)$$

Substitute and solve for A:

$$A = \sin^{-1}\left(\frac{7 \cdot \sin 40°}{5.79}\right)$$

better to keep as exact value

$$\therefore A = 51°$$

c) $B = 180 - 40° - 51°$
$= 89°$

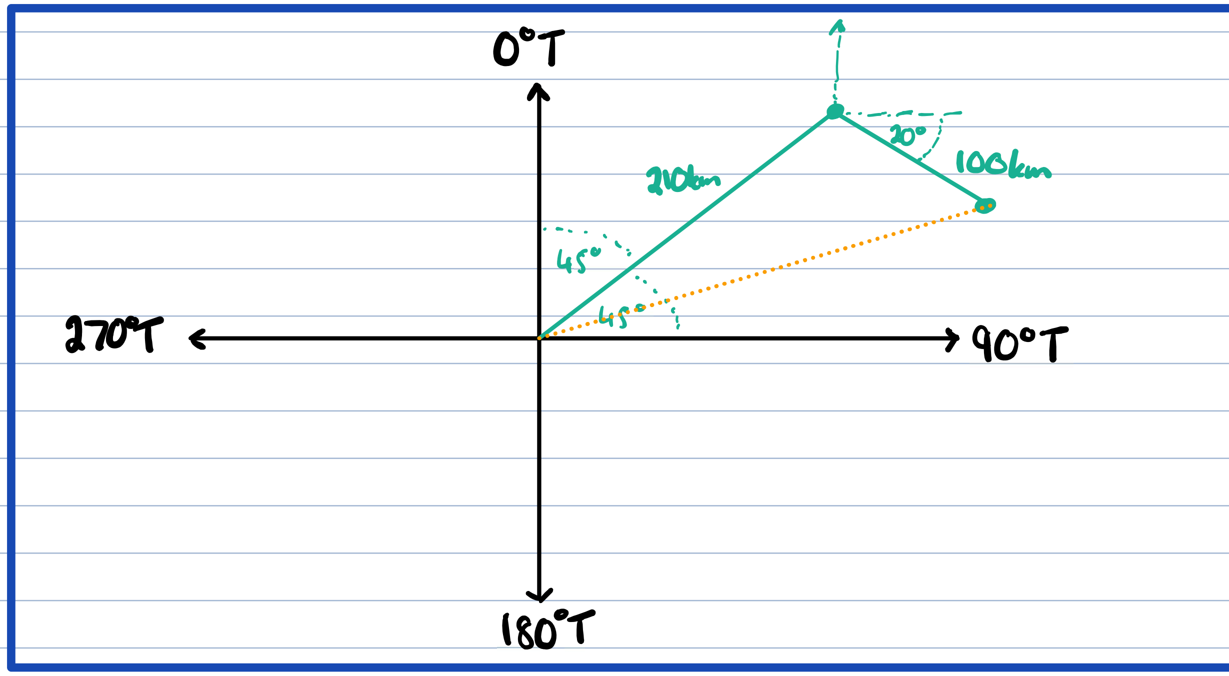
e) Area $= \frac{1}{2} ab \cdot \sin C$

$= \frac{1}{2}(7)(9) \cdot \sin 40°$

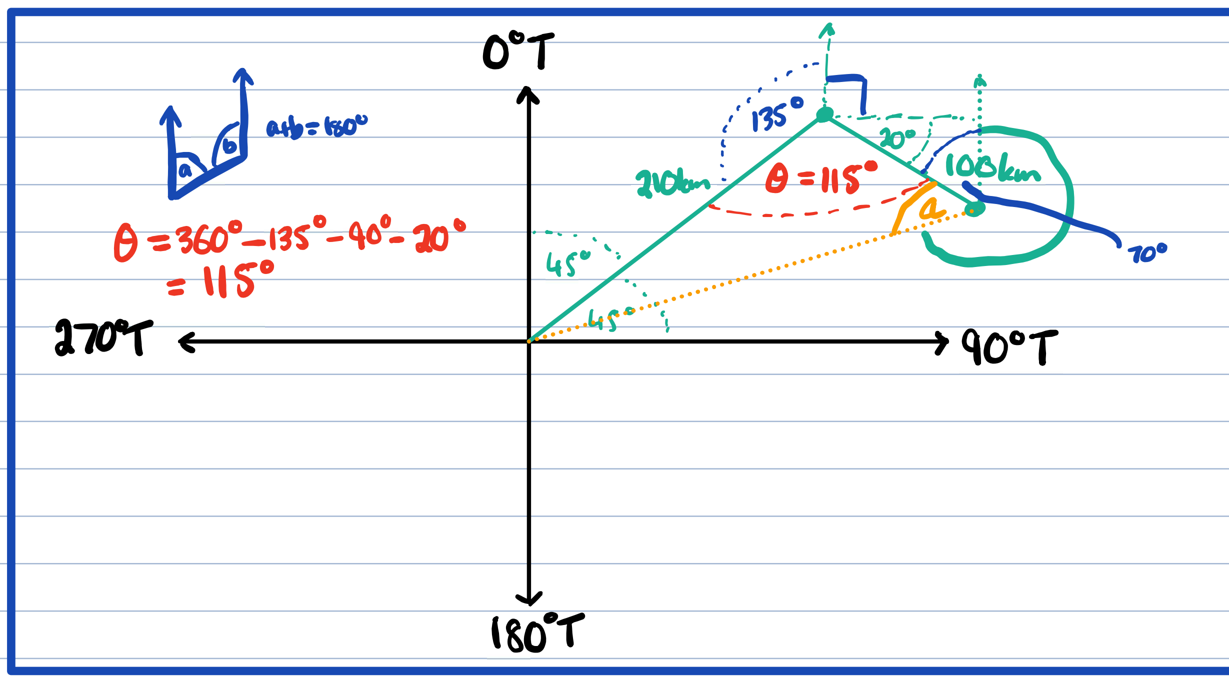
$= 20.25 \text{ cm}^2$

Q4) a)

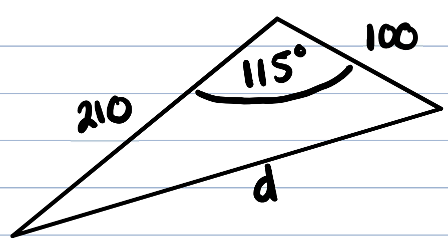
original diagram




Determining missing angles




b)




$$d = \sqrt{210^2 + 100^2 - 2 \times 210 \times 100 \times \cos 115°}$$
$$= 268.05 \text{ km}$$

c) Step 1: Use sine rule to find a :

$$\frac{\sin a}{210} = \frac{\sin 115°}{268.05}$$

$$a = 45.24°$$

Step 2: Subtract 70° and 45.24° from 360° to find bearing back to start point :

$$\text{Bearing} = 360° - 70° - 45.24°$$
$$= 244.76°T$$

# Unit 4 Topic 3 :

Q1) $Pr(\text{success}) = 0.88$
$Pr(\text{failure}) = 1 - 0.88$
$= 0.12$

b) $0.12$

c) mean $= np$
$= 1 \times 0.88$
$= 0.88$

Var $= npq$
$= 0.88 \times 0.12$
$= 0.1056$

Q2) $p = 0.75$, $n = 5$, $q = 0.25$

a)
Using binomial probability feature (Pdf) on graphics calculator:

$$Pr(X=4) = 0.3955$$

b) Using binomial probability feature (Cdf) on graphics calculator:

$$Pr(X \geq 4) = 0.6328$$

c) mean $= np$
$= 5 \times 0.75$
$= 3.75$

var $= npq$
$= 3.75 \times 0.25$
$= 0.9375$

$\sigma = \sqrt{Var}$
$= \sqrt{0.9375}$
$\approx 0.9682$

Q3) a) $p = 0.5$, $q = 0.5$, $n = 3$

(i) $Pr(3 \text{ heads}) = \binom{3}{3} (0.5)^3 (0.5)^0$

$= 1 \cdot (0.5)^3 \cdot 1$
$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
$= \frac{1}{8}$ or $0.125$

(ii) $Pr(\text{At least 1 head}) = 1 - Pr(\text{No heads})$
$= 1 - \binom{3}{0} (0.5)^0 (0.5)^3$
$= 1 - \frac{1}{8}$
$= \frac{7}{8}$

Remember:
$\binom{3}{3} = 1$ and $\binom{3}{0} = 1$

$\binom{3}{1} = 3$ and $\binom{3}{2} = 3$

These are important tech free skills!

Q3b) $p = 1/4, \quad q = 3/4, \quad n = 2$

(i) $Pr(1 \text{ blue}) = \binom{2}{1} (1/4)^1 (3/4)^1$

$= 2 \cdot 1/4 \cdot 3/4$
$= 2/4 \cdot 3/4$
$= 6/16$
$= 3/8$

(ii) $Pr(\text{at least one blue}) = 1 - Pr(0 \text{ blue})$

$= 1 - \binom{2}{0} (1/4)^0 (3/4)^2$

$= 1 - 1 \cdot 1 \cdot 9/16$
$= 1 - 9/16$
$= 16/16 - 9/16$
$= 7/16$

Q4) a) $p = 0.6, q = 0.4, n = 8$

(i) Using binomial probability feature (Cdf) on graphics calculator:

$Pr(X \geq 3) = 0.9502$

(ii) Using binomial probability feature (Cdf) on graphics calculator:

$Pr(X \leq 5) = 0.6846$

(iii) Using binomial probability feature (Cdf) on graphics calculator:

$Pr(3 \leq X \leq 5) = 0.6348$

Q5) $p = 0.05, \quad q = 0.95, \quad n = 20$

a) Using binomial probability feature (Pdf) on graphics calculator:

$Pr(X = 2) = 0.1887$

b) Using binomial probability feature (Cdf) on graphics calculator:

$Pr(X \leq 2) = 0.9245$

# Unit 4 Topic 4 :

Q1) For PDF : $\int_{-\infty}^{\infty} p(x)\,dx = 1$

a)
$$1 = \int_{-1}^{1} k(1-x^2)\,dx$$

$$1 = k\int_{-1}^{1} 1-x^2 \quad dx$$

$$1 = k\cdot\left[ x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$\frac{1}{k} = \left(1 - \frac{(1)^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right)$$

$$\frac{1}{k} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$

$-\frac{3}{3} + \frac{1}{3}$

$= -\frac{2}{3}$

$$\frac{1}{k} = \frac{2}{3} - -\frac{2}{3}$$

$$\frac{1}{k} = \frac{4}{3}$$

$$\therefore k = \frac{3}{4}$$

b) Using Formula Sheet + Graphics Calculator :

$$E(x) = \int_{-1}^{1} x\cdot\frac{3}{4}(1-x^2)\,dx$$

Using GDC:

$$E(x) = \mu = 0$$

c) Using Formula Sheet + Graphics Calculator :

$$Var(X) = \int_{-\infty}^{\infty}(x-\mu)^2\,p(x)\,dx$$

$$= \int_{-1}^{1}(x-0)^2\cdot\frac{3}{4}(1-x^2)\,dx$$

$$= \frac{1}{5}$$

d) Using GDC:

$$Pr(X > 0.5) = \int_{0.5}^{1}\frac{3}{4}(1-x^2)\,dx = 0.15625$$

e)

$$Pr(-0.5 < X < 1) = \int_{-0.5}^{1}\frac{3}{4}\cdot(1-x^2)\,dx = 0.84375$$

Q2) $f(x) = 0.1e^{-0.1x} \quad x \geq 0$

a) $E(x) = \lim_{k\to\infty} \int_0^k x \cdot 0.1e^{-0.1x}\,dx$

Using Graphics Calculator with $k = 10^{10}$

$$E(x) = 10$$

Using Graphics calculator + formula sheet:

$$Var(x) = 100$$

$$\begin{aligned}\sigma &= \sqrt{Var(x)} \\ &= \sqrt{100} \\ &= 10\end{aligned}$$

b) $Pr(1 < x < 2) = \int_1^2 0.1e^{-0.1x}\,dx = 0.0861$

Q3)

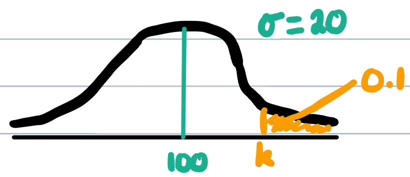
a) $\mu = 100, \ \sigma = 20$

$$z = \frac{x-\mu}{\sigma}$$

$$z = \frac{130-100}{20} = \frac{30}{20}$$

$$\therefore z = 1.5$$

b)



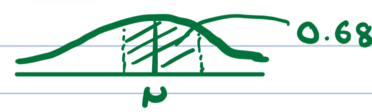

Using Inverse Normal feature:

$$\therefore k = 125.631$$

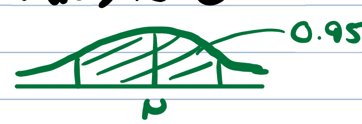
c) Using normal Cdf feature:

$Pr(72 < x < 115) = 0.6926$

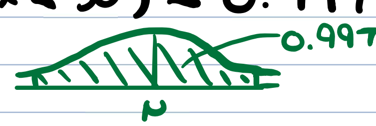
d) $Pr(-1\sigma < x < 1\sigma) \approx 0.68 \approx 68\%$



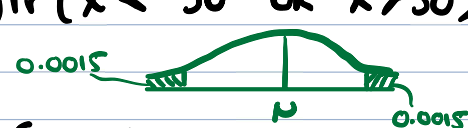

e) $Pr(-2\sigma < x < 2\sigma) \approx 0.95 \approx 95\%$




f) $Pr(-3\sigma < x < 3\sigma) \approx 0.997 \approx 99.7\%$

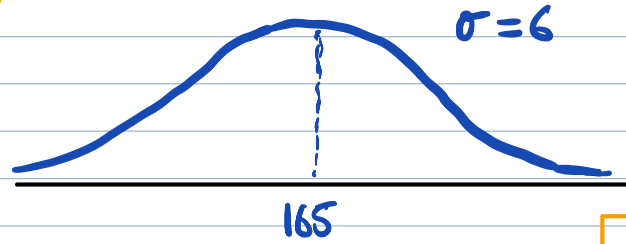



g) $Pr(x < -3\sigma \text{ OR } x > 3\sigma) \approx 1 - 0.997 \approx 0.003 \approx 0.3\%$

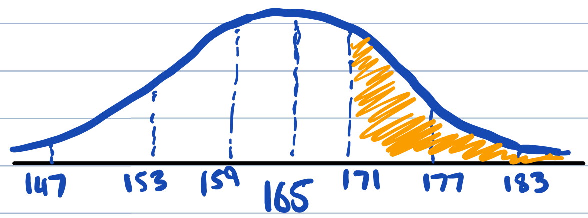



h) See above

Q4) Always good to draw a diagram for normal distribution questions

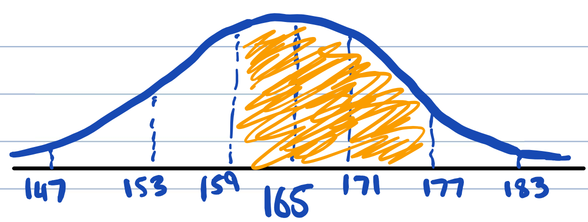



a)

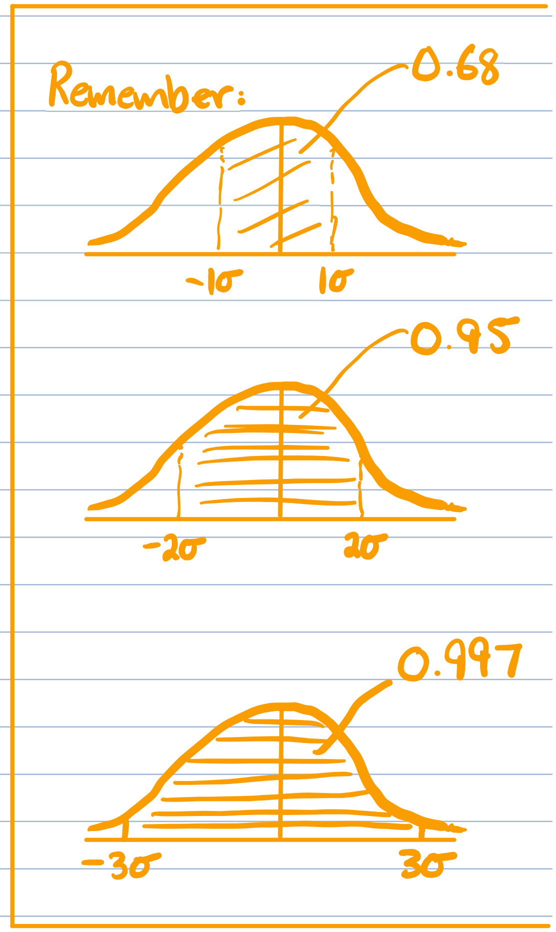



$Pr(X > 171) = 1 - 0.5 - 0.34$
$= 0.16$

b)




$Pr(159 < x < 177) = 0.34 + 0.475$
$= 0.815$




c) Using normal cdf feature:

$Pr(160 < x < 180) = 0.7915$
$= 79.15\%$

d) Using normal cdf feature:

$Pr(X < 140) = 0.000015$
$= 0.0015\%$

e) Using inverse normal feature:
Bottom 25% = 160.95 cm or less

f) Using inverse normal feature:
Top 1% = 178.96 cm or more

# Unit 4 Topic 5 :

Q1)

a) $\hat{p} = 72/120 = 36/60 = 6/10 = 3/5$

$\boxed{\therefore \hat{p} = 3/5 \quad \text{or} \quad 0.6}$

b) Using formula sheet:

mean $= p = 0.65$

$\sigma = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.65 \times 0.35}{120}} \approx 0.0435$

Q2) Using graphics calculator:

$n = 500$, $x = 310$, 95% CI $(z = 1.96)$

Confidence interval using GDC:

$(0.5775, 0.6625)$

b) We can be 95% confident that the population proportion is between 57.75% and 66.25%.

Q3) Using graphics calculator:

$n = 150$, $x = 65$, 90% CI $(z = 1.65)$

Confidence interval using GDC:

$(0.3668, 0.4999)$

b) We can be 90% confident that the population proportion is between 36.68% and 49.99%.

Q4) Using Formula sheet :

$$ME = z\sqrt{\frac{pq}{n}}$$

where ME is margin of error

$$0.04 = 1.96\sqrt{\frac{0.55\times0.45}{n}}$$

95% CI → z = 1.96

Re-arrange for n :

$$\left(\frac{0.04}{1.96}\right)^2 = \frac{0.2475}{n}$$

$$n = \frac{0.2475}{\left(\frac{0.04}{1.96}\right)^2}$$

$$n = 594.25$$

∴ A sample size of 595 is required to achieve a margin of error of 4%.

b) A larger margin of error means the width is of the confidence interval will become wider

c) As the sample size increases, the width of the confidence interval becomes more narrow (e.g. more precise)

Q5) Using same approach as Q4a :

99% CI → z = 2.58

$$0.025 = 2.58\sqrt{\frac{0.6\times0.4}{n}}$$

Rearrange for n :

$$\therefore n = 2556.06$$

∴ A sample size of 2557 is required to achieve a margin of error of 2.5%.